ELSEVIER

International Journal of Solids and Structures 41 (2004) 3225–3242

INTERNATIONAL JOURNAL OF SOLIDS and STRUCTURES

www.elsevier.com/locate/ijsolstr

An accurate method for transcendental eigenproblems with a new criterion for eigenfrequencies

Zhaohui Qi [a], D. Kennedy [b,*], F.W. Williams [c]

[a] *Department of Engineering Mechanics, Dalian University of Technology, Dalian 116023, PR China*
[b] *Cardiff School of Engineering, Cardiff University, P.O. Box 925, The Parade, Cardiff CF24 0YF, Wales, UK*
[c] *Department of Building and Construction, City University of Hong Kong, Kowloon, Hong Kong*

Received 28 April 2003; received in revised form 6 January 2004

Available online 3 March 2004

Abstract

A new computational method for the transcendental eigensolution of structural dynamics is proposed. It is based on a new criterion for detecting the eigenfrequency. Instead of trying to find the roots of the determinant of the dynamic stiffness matrix, this method seeks the frequency that makes the last energy norm vanish, so matching the physical definition of eigenfrequency except for those eigenfrequencies for which the last degree of freedom is nodal in the mode, for which the method is suitably modified. By using the property of derivatives of energy norms, the eigenproblem is transformed safely into a specific initial value problem of an ordinary differential equation. Among many available methods to solve the resulting ordinary differential equation, the one-step Runge–Kutta method is proved to be a simple and efficient way to obtain eigensolutions, as confirmed by a numerical example.
© 2004 Elsevier Ltd. All rights reserved.

Keywords: Structures; Vibration; Eigenvalues; Mode shape; Wittrick–Williams algorithm

1. Introduction

Transcendental eigenproblems arise in structural dynamics when the exact dynamic stiffness matrix, the coefficients of which are transcendental functions of frequency, is used instead of the approximate one of finite element (FEM) analysis, which gives a linear eigenproblem (Blaszkowiak and Kaczkowski, 1966; Williams and Wittrick, 1970). Therefore they are very different from the usual linear eigenvalue problems, because they possess an infinite number of eigensolutions even though the order of the dynamic stiffness matrix is finite. Hence the higher natural frequencies are obtainable from the transcendental eigenproblem, unlike the linear eigenproblem in which higher natural frequencies can be modelled with reasonable accuracy only by accepting the large computational cost of making the size of elements small enough (Hager and Wiberg, 1999; Wiberg et al., 1999).

* Corresponding author. Tel.: +44-29-2087-5340; fax: +44-29-2087-4826.
E-mail address: kennedyd@cf.ac.uk (D. Kennedy).

0020-7683/$ - see front matter © 2004 Elsevier Ltd. All rights reserved.
doi:10.1016/j.ijsolstr.2004.01.022

In contrast to the many well-established methods for solving linear eigenproblems (Bathe, 1996; Meirovitch, 1980; Collar and Simpson, 1987), there are only a few published methods available for solving transcendental eigenproblems. Based on the Sturm sequence property of eigenproblems, Wittrick and Williams (1971) developed an algorithm which enables one to count the number of eigenfrequencies exceeded by a given trial frequency. Hence upper and lower bounds on any specified eigenfrequency are easily determined. Moreover, the upper and lower bounds can always be made to approach each other by the bisection method, i.e. by recursively selecting a trial value half way between the current upper and lower bounds, which itself becomes the next upper or lower bound (Williams and Wittrick, 1970, 1983; Wittrick and Williams, 1971, 1973; Williams and Anderson, 1986; Ye and Williams, 1995). This is regarded as a reliable method for solving transcendental eigenproblems and was the first way in which the Wittrick–Williams algorithm was applied.

Unfortunately, the computation of the modes of vibration does not benefit directly from the Wittrick–Williams algorithm, in the sense that the algorithm can be used to bound an eigenvalue but does not give bounds on the associated eigenvector, i.e. the mode. Thus when using the Wittrick–Williams algorithm the commonest existing procedure is to converge on the eigenfrequency first and then to find the mode by, in effect, inverting a matrix which is not usually very well-conditioned (Hopper and Williams, 1977; Howson, 1979; Howson et al., 1983; Ronagh et al., 1995; Williams and Yuan, 1999). As a result, the mode of vibration is usually not very accurate, typically having two or three fewer significant figures of accuracy than that to which the eigenvalue has been found.

Although the bisection method often employed when using the Wittrick–Williams algorithm shares the reliability of the algorithm itself, it possesses only linear order convergence. Alternatives to bisection have been developed to accelerate convergence when using the Wittrick–Williams algorithm (Simpson, 1984; Williams and Kennedy, 1988; Kennedy and Williams, 1991) but further work is needed in this area.

In this paper, the foundation of the Wittrick–Williams algorithm is reviewed, with the emphasis put on reducing the dynamic stiffness matrix to diagonal form by using a congruent transformation. The physical meanings of the matrix of the congruent transformation and of the resulting diagonal matrix are fully developed. For example, the last column of the congruent transformation matrix is the mode of displacement when the forces acting at all the degrees of freedom except the last one are zero, while the force acting at the last degree of freedom is equal to the last element of the diagonal matrix. This last element is very important to the computation of eigensolutions and is called the last energy norm. Based on the physical definitions of the eigenfrequency and its corresponding mode of vibration, it is easy to understand that usually if the last energy norm tends to zero, then the last column of the congruent matrix tends to the mode of vibration and also the trial frequency that makes this last energy norm vanish is an eigenfrequency of the structure. Therefore, the mode of vibration can be found without inversion of any matrix at the same time as the associated eigenfrequency is obtained. Using the qualitative properties of the last energy norm, the differential of the trial frequency with respect to it is found and the transcendental eigenvalue problem is transformed into an initial value problem of an ordinary differential equation. Consequently, any advanced methods for solving ordinary differential equations can be used to solve the transcendental eigenproblem.

2. Review of the Wittrick–Williams algorithm

A structure is usually discretized by using FEM in order to obtain the numerical solution of the problem whenever analytical solution is impossible or inconvenient. In this FEM case, the eigenproblem can be stated mathematically as being to find a positive real parameter λ in the equation

$$(\mathbf{K} - \lambda\mathbf{M})\mathbf{x} = \mathbf{0}, \tag{1}$$

which makes the vector $\mathbf{x}$ non-trivial, where $\mathbf{K}$ and $\mathbf{M}$ are the $(n \times n)$ static stiffness and mass matrices, respectively, and both of these matrices are real, symmetric and non-negative definite. λ is related to the circular frequency ω by $\lambda = \omega^2$. It is well known that there is a non-singular matrix $\mathbf{X}$ whose ith column is the eigenvector $\mathbf{x}_i$ associated with the eigenvalue λ_i, such that, for any trial value $\bar{\lambda}$,

$$\mathbf{X}^{\mathrm{T}}(\mathbf{K} - \bar{\lambda}\mathbf{M})\mathbf{X} = \mathrm{Diag}(\lambda_1 - \bar{\lambda}, \ldots, \lambda_i - \bar{\lambda}, \ldots, \lambda_n - \bar{\lambda}). \tag{2}$$

If $\bar{\lambda}$ is not coincident with any of the eigenvalues, the dynamic stiffness matrix $(\mathbf{K} - \bar{\lambda}\mathbf{M})$ can be factorized into

$$(\mathbf{K} - \bar{\lambda}\mathbf{M}) = \mathbf{L}\mathbf{U} = \mathbf{L}\mathbf{D}\mathbf{L}^{\mathrm{T}} \tag{3}$$

by the process of Gauss elimination, where $\mathbf{L}$ is a non-singular lower triangular matrix with unit diagonal elements, $\mathbf{U}$ is an upper triangular matrix, and $\mathbf{D}$ is a diagonal matrix. From Eq. (3), it can easily be seen that

$$\mathbf{L}^{-1}(\mathbf{K} - \bar{\lambda}\mathbf{M})\mathbf{L}^{-\mathrm{T}} = \mathbf{D}. \tag{4}$$

Based on Sylvester's law of inertia (Mirsky, 1982), if $\mathbf{D}_1$ and $\mathbf{D}_2$ are real diagonal matrices obtained from the same real symmetric matrix $\mathbf{A}$ by two congruent transformations, e.g. $\mathbf{P}_1^{\mathrm{T}}\mathbf{A}\mathbf{P}_1 = \mathbf{D}_1$ and $\mathbf{P}_2^{\mathrm{T}}\mathbf{A}\mathbf{P}_2 = \mathbf{D}_2$, where $\mathbf{P}_1$ and $\mathbf{P}_2$ are real non-singular matrices, then $\mathbf{D}_1$ and $\mathbf{D}_2$ must have the same number of negative diagonal elements. Therefore this number is invariant under the congruent transformation; it is called the sign count of $\mathbf{A}$ and is denoted by $s\{\mathbf{A}\}$. Now if the number of negative diagonal elements of the matrix on the right-hand side of Eq. (2) is $\bar{J}(\bar{\lambda})$, it is clearly the number of eigenvalues of the structure exceeded by $\bar{\lambda}$. Hence comparing Eqs. (2) and (4) gives

$$\bar{J}(\bar{\lambda}) = s\{\mathbf{K} - \bar{\lambda}\mathbf{M}\}. \tag{5}$$

The eigenvalues of Eq. (1) are only approximate eigenvalues of the structure because of the errors arising from the discretization process. An obvious inconsistency of any discrete model is that it predicts only a finite number of eigenfrequencies of the structure, whereas the real structure has an unlimited number. Therefore if a high eigenfrequency is of interest, smaller elements have to be employed, which usually results in the matrices $\mathbf{K}$ and $\mathbf{M}$ being much larger and hence in a greatly increased solution time.

Fortunately, there are always ways to reduce the number of governing equations. For instance, if $\mathbf{x}$ is divided into two parts, namely $\mathbf{x}_c$, the visible vector of order m, and $\mathbf{x}_i$, the invisible vector of order $(n - m)$, Eq. (1) can be re-written as

$$\begin{bmatrix} (\mathbf{K}_{ii} - \bar{\lambda}\mathbf{M}_{ii}) & (\mathbf{K}_{ic} - \bar{\lambda}\mathbf{M}_{ic}) \\ (\mathbf{K}_{ci} - \bar{\lambda}\mathbf{M}_{ci}) & (\mathbf{K}_{cc} - \bar{\lambda}\mathbf{M}_{cc}) \end{bmatrix} \begin{bmatrix} \mathbf{x}_i \\ \mathbf{x}_c \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix} \tag{6}$$

and $\mathbf{x}_i$ can be expressed in terms of $\mathbf{x}_c$ as

$$\mathbf{x}_i = -(\mathbf{K}_{ii} - \bar{\lambda}\mathbf{M}_{ii})^{-1}(\mathbf{K}_{ic} - \bar{\lambda}\mathbf{M}_{ic})\mathbf{x}_c. \tag{7}$$

Now $\mathbf{x}_c$ can be obtained by solving the equation

$$\overline{\mathbf{K}}_{\mathrm{D}}(\bar{\lambda})\mathbf{x}_c = \mathbf{0}, \tag{8}$$

where the reduced dynamic stiffness matrix $\overline{\mathbf{K}}_{\mathrm{D}}(\bar{\lambda})$ is given by

$$\overline{\mathbf{K}}_{\mathrm{D}}(\bar{\lambda}) = (\mathbf{K}_{cc} - \bar{\lambda}\mathbf{M}_{cc}) - (\mathbf{K}_{ci} - \bar{\lambda}\mathbf{M}_{ci})(\mathbf{K}_{ii} - \bar{\lambda}\mathbf{M}_{ii})^{-1}(\mathbf{K}_{ic} - \bar{\lambda}\mathbf{M}_{ic}). \tag{9}$$

When the dynamic stiffness matrix is subjected to a congruent transformation for which

$$\mathbf{P} = \begin{bmatrix} \mathbf{I}_{n-m} & -(\mathbf{K}_{ii} - \bar{\lambda}\mathbf{M}_{ii})^{-1}(\mathbf{K}_{ic} - \bar{\lambda}\mathbf{M}_{ic}) \\ \mathbf{0} & \mathbf{I}_m \end{bmatrix}, \tag{10}$$

where $\mathbf{I}_k$ is the unit matrix of order k, it reduces to the following matrix with quasi-diagonal form

$$\mathbf{P}^{\mathrm{T}}\begin{bmatrix}(\mathbf{K}_{ii}-\bar{\lambda}\mathbf{M}_{ii}) & (\mathbf{K}_{ic}-\bar{\lambda}\mathbf{M}_{ic})\\ (\mathbf{K}_{ci}-\bar{\lambda}\mathbf{M}_{ci}) & (\mathbf{K}_{cc}-\bar{\lambda}\mathbf{M}_{cc})\end{bmatrix}\mathbf{P}=\begin{bmatrix}(\mathbf{K}_{ii}-\bar{\lambda}\mathbf{M}_{ii}) & \mathbf{0}\\ \mathbf{0} & \overline{\mathbf{K}}_{\mathrm{D}}(\bar{\lambda})\end{bmatrix}. \tag{11}$$

It is obvious that each non-null sub-matrix on the right-hand side of Eq. (11) can be further reduced to a diagonal matrix separately by a congruent transformation. Therefore, the sign count of the dynamic stiffness matrix is the sum of the sign counts of these sub-matrices, i.e.

$$\bar{J}(\bar{\lambda})=s\{\mathbf{K}_{ii}-\bar{\lambda}\mathbf{M}_{ii}\}+s\{\overline{\mathbf{K}}_{\mathrm{D}}(\bar{\lambda})\}. \tag{12}$$

Now, suppose that the whole structure is composed of sub-structures, so that the nodes of the structure can be divided into two groups according to whether or not they are on the boundary of any sub-structure. A node that belongs to such a boundary is called a sub-boundary node, and a node that is geometrically at an inner point of a sub-structure is called an inner node. For example, for the structure shown in Fig. 1 the nodes C_1, C_2, C_3 and C_4 are sub-boundary nodes, and the nodes denoted by hollow circles are inner nodes. Note that in this simple example each sub-structure represents a single member, but that in general sub-structures can consist of many members and such complicated sub-structures are included by the theory presented in this paper.

Because inner nodes can only belong to one sub-structure, the usual FEM procedure for assembling the global dynamic stiffness matrix shows that, if all the components of $\mathbf{x}_i$ are ordered such that the degrees of freedom of the inner nodes of the first sub-structure precede those of the second sub-structure which in turn precede those of the third sub-structure, etc., then the matrix $(\mathbf{K}_{ii}-\bar{\lambda}\mathbf{M}_{ii})$ must be quasi-diagonal, i.e. it has the form

$$(\mathbf{K}_{ii}-\bar{\lambda}\mathbf{M}_{ii})=\begin{bmatrix}(\mathbf{K}_1-\bar{\lambda}\mathbf{M}_1) & & & & \\ & (\mathbf{K}_2-\bar{\lambda}\mathbf{M}_2) & & & \\ & & \ddots & & \\ & & & (\mathbf{K}_{l-1}-\bar{\lambda}\mathbf{M}_{l-1}) & \\ & & & & (\mathbf{K}_l-\bar{\lambda}\mathbf{M}_l)\end{bmatrix} \tag{13}$$

where $(\mathbf{K}_k-\bar{\lambda}\mathbf{M}_k)$ is the approximate FEM dynamic stiffness matrix for the kth sub-structure with its boundary clamped, and l is equal to n_{s}, the total number of sub-structures. As the reduction of any sub-

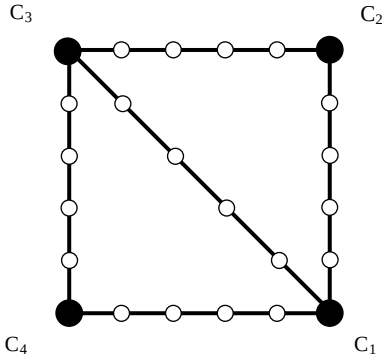

Fig. 1. Definition of sub-boundary and inner nodes of a structure. Solid circles and hollow circles indicate sub-boundary and inner nodes, respectively.

matrix in Eq. (13) to a diagonal matrix can be performed separately, the sign count of the matrix on the left-hand side of Eq. (13) is the sum of the sign counts of the sub-matrices on the right-hand side, i.e.

$$s\{\mathbf{K}_{ii} - \bar{\lambda}\mathbf{M}_{ii}\} = \sum_{k=1}^{n_s} s\{\mathbf{K}_k - \bar{\lambda}\mathbf{M}_k\}. \tag{14}$$

As the number of inner nodes in each sub-structure tends towards infinity, each of the matrices $(\mathbf{K}_k - \bar{\lambda}\mathbf{M}_k)$ $(k = 1, 2, \ldots, n_s)$ tends to the exact dynamic stiffness matrix of the kth sub-structure with its boundary clamped, and Eq. (8) tends to

$$\mathbf{K}_D(\bar{\lambda})\mathbf{x}_c = \mathbf{0}, \tag{15}$$

i.e. $\overline{\mathbf{K}}_D(\bar{\lambda})$ approaches $\mathbf{K}_D(\bar{\lambda})$, the exact dynamic stiffness matrix of the structure with respect to the visible vector of variables, which is a transcendental function of $\bar{\lambda}$. Therefore Eqs. (12) and (14) give

$$J(\bar{\lambda}) = \sum_{k=1}^{n_s} J_k(\bar{\lambda}) + s\{\mathbf{K}_D(\bar{\lambda})\}, \tag{16}$$

where $J(\bar{\lambda})$ is the number of eigenvalues of the structure that are exceeded by the trial value $\bar{\lambda}$ and $J_k(\bar{\lambda})$ is the number of eigenvalues of the kth sub-structure with its boundary clamped that are exceeded by $\bar{\lambda}$.

As shown by Eq. (9), $\mathbf{K}_D(\bar{\lambda})$ is no longer a linear function of $\bar{\lambda}$, which is why finding the eigenfrequencies and the modes of vibration via Eq. (15) is called a transcendental eigenproblem.

Eq. (16) is the foundation of the Wittrick–Williams algorithm (Wittrick and Williams, 1971). It can be seen that its derivation is based on the congruent transformation of the dynamic stiffness matrix, instead of on the Sturm sequence property which was the starting point for earlier proofs of the algorithm. It will now be shown that the congruent transformation plays a very important role in the solution of the transcendental eigenproblem.

3. Modes of displacements for reduction of the dynamic stiffness matrix

In order to obtain $s\{\mathbf{K}_D(\bar{\lambda})\}$, a diagonal matrix $\mathbf{D}$ that is congruent to $\mathbf{K}_D(\bar{\lambda})$ is desired. From Eq. (4), it is seen that there is a unique upper triangular matrix $\mathbf{P} = \mathbf{L}^{-T}$ that can reduce the dynamic stiffness matrix to diagonal form, i.e. for which

$$\mathbf{P}^T\mathbf{K}_D(\bar{\lambda})\mathbf{P} = \mathbf{D}. \tag{17}$$

Each column of $\mathbf{P}$ can be physically interpreted as a possible mode of displacements of the structure, so that the corresponding column of the matrix $\mathbf{F} = \mathbf{K}_D(\bar{\lambda})\mathbf{P}$ comprises the forces causing these displacements. Therefore the elements of $\mathbf{D}$ are called the energy norms of the modes of displacements. Now pre-multiplying Eq. (17) by $\mathbf{P}^{-T}$ enables $\mathbf{F}$ to be re-written as

$$\mathbf{F} = \mathbf{P}^{-T}\mathbf{D} \tag{18}$$

which shows clearly that $\mathbf{F}$ is a lower triangular matrix. Therefore, see Fig. 2(a), the first mode of displacements $\mathbf{p}_1$ is obtained by applying a force f_{11} at the first degree of freedom so that $p_{11} = 1$ (p_{ij} indicates the ith component of $\mathbf{p}_j$, the jth column of $\mathbf{P}$, and f_{ij} indicates the ith component of $\mathbf{f}_j$, the jth column of $\mathbf{F}$) and fixing the remaining degrees of freedom. In general, the jth mode of displacements $\mathbf{p}_j$ is produced by applying a force f_{jj} at the jth degree of freedom (so that $p_{jj} = 1$), fixing the $(j+1)$th, $(j+2)$th, …, nth degrees of freedom, and applying no forces at the 1st, 2nd, …, $(j-1)$th degrees of freedom, as illustrated by Fig. 2(b)–(d).

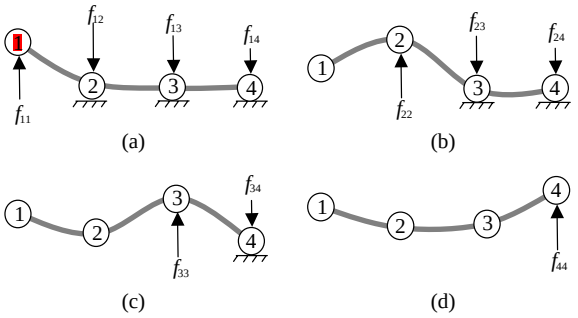

Fig. 2. Modes of displacements and forces for the congruent transformation of the dynamic stiffness matrix.

Although the modes of displacements can be evaluated by the triangular decomposition of the dynamic stiffness matrix as shown in Eq. (4), they can instead be obtained recursively by

$$\mathbf{p}_1 = \mathbf{e}_1 \tag{19}$$

$$\mathbf{f}_1 = \mathbf{k}_1 \tag{20}$$

$$\mathbf{p}_k = \mathbf{e}_k - \sum_{i=1}^{k-1} \frac{f_{ik}}{f_{ii}} \mathbf{p}_i \tag{21}$$

$$\mathbf{f}_k = \mathbf{k}_k - \sum_{i=1}^{k-1} \frac{f_{ik}}{f_{ii}} \mathbf{f}_i \tag{22}$$

where $\mathbf{k}_k$ is the kth column of $\mathbf{K}_{\mathrm{D}}(\bar{\lambda})$ and $\mathbf{e}_k$ is the kth column of the unit matrix of order n.

d_i, the ith diagonal element of $\mathbf{D}$, can be interpreted as the necessary input energy to produce the mode of displacements $\mathbf{p}_i$. As an immediate consequence of Eq. (17), this can be expressed in terms of the forces as

$$d_i = f_{ii}. \tag{23}$$

Moreover, if the ith principal minor determinant of $\mathbf{K}_{\mathrm{D}}(\bar{\lambda})$ is D_i, then

$$d_i = \frac{D_i}{D_{i-1}}. \tag{24}$$

This result can be obtained by writing Eq. (17) in the sub-matrix form

$$\begin{bmatrix} \mathbf{P}^{11} & \mathbf{P}^{12} \\ \mathbf{0} & \mathbf{P}^{22} \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \mathbf{K}_{\mathrm{D}}^{11} & \mathbf{K}_{\mathrm{D}}^{12} \\ \mathbf{K}_{\mathrm{D}}^{21} & \mathbf{K}_{\mathrm{D}}^{22} \end{bmatrix} \begin{bmatrix} \mathbf{P}^{11} & \mathbf{P}^{12} \\ \mathbf{0} & \mathbf{P}^{22} \end{bmatrix} = \begin{bmatrix} \mathbf{D}^{11} & \mathbf{0} \\ \mathbf{0} & \mathbf{D}^{22} \end{bmatrix}, \tag{25}$$

from which it is easily seen that

$$\mathbf{D}^{11} = \mathbf{P}^{11\mathrm{T}} \mathbf{K}_{\mathrm{D}}^{11} \mathbf{P}^{11}. \tag{26}$$

Eq. (24) follows immediately from the fact that the determinant of $\mathbf{P}^{11}$ is unity.

4. Curves of energy norms

According to Eq. (17), the energy norms can be written as

$$d_i = \mathbf{p}_i^{\mathrm{T}} \mathbf{K}_{\mathrm{D}}(\bar{\lambda}) \mathbf{p}_i \quad (i = 1, \ldots, n). \tag{27}$$

If the order of the degrees of freedom does not change during the process of the evaluation of the energy norms, each d_i is a definite function of the trial value $\bar{\lambda}$. The derivative of the function with respect to $\bar{\lambda}$ can be expressed as

$$\frac{\mathrm{d}d_i}{\mathrm{d}\bar{\lambda}} = 2\frac{\mathrm{d}\mathbf{p}_i^{\mathrm{T}}}{\mathrm{d}\bar{\lambda}}\mathbf{f}_i + \mathbf{p}_i^{\mathrm{T}}\frac{\mathrm{d}\mathbf{K}_{\mathrm{D}}(\bar{\lambda})}{\mathrm{d}\bar{\lambda}}\mathbf{p}_i \quad (i = 1, \ldots, n). \tag{28}$$

The following equalities

$$\left.\begin{array}{l} p_{ii} = 1 \\ p_{ik} = 0 \quad (k > i) \\ f_{ik} = 0 \quad (k < i) \end{array}\right\} \tag{29}$$

are true for any trial frequency, where p_{ik} is the kth component of the ith mode of displacements, so that

$$\frac{\mathrm{d}\mathbf{p}_i^{\mathrm{T}}}{\mathrm{d}\bar{\lambda}}\mathbf{f}_i = \sum_{k=1}^{i}\frac{\mathrm{d}p_{ik}^{\mathrm{T}}}{\mathrm{d}\bar{\lambda}}f_{ik} = \frac{\mathrm{d}p_{ii}}{\mathrm{d}\bar{\lambda}}f_{ii} = 0. \tag{30}$$

In other words, the first term in the right-hand side of Eq. (28) makes no contribution to the derivative, i.e.

$$\frac{\mathrm{d}d_i}{\mathrm{d}\bar{\lambda}} = \mathbf{p}_i^{\mathrm{T}}\frac{\mathrm{d}\mathbf{K}_{\mathrm{D}}(\bar{\lambda})}{\mathrm{d}\bar{\lambda}}\mathbf{p}_i \quad (i = 1, \ldots, n). \tag{31}$$

If the structure is discretized by using finite elements, the dynamic stiffness matrix is a linear function of $\bar{\lambda}$. In this case, Eq. (31) can be simplified as

$$\frac{\mathrm{d}\bar{d}_i}{\mathrm{d}\bar{\lambda}} = \mathbf{p}_i^{\mathrm{T}}\frac{\mathrm{d}(\mathbf{K} - \bar{\lambda}\mathbf{M})}{\mathrm{d}\bar{\lambda}}\mathbf{p}_i = -\mathbf{p}_i^{\mathrm{T}}\mathbf{M}\mathbf{p}_i \quad (i = 1, \ldots, m, m+1, \ldots, m+n), \tag{32}$$

where m is the number of degrees of freedom corresponding to the inner nodes of the structure and the $(m+k)$th degree of freedom in the finite element model is associated with the kth degree of freedom in the exact model. As the mass matrix $\mathbf{M}$ is positive definite, Eq. (32) shows that the derivative of each $\bar{d}_i$ is negative. When m tends to infinity, $\bar{d}_{m+i}$ tends to d_i. Therefore, the derivatives of the energy norms are negative, i.e.

$$\frac{\mathrm{d}d_i}{\mathrm{d}\bar{\lambda}} < 0 \quad (i = 1, \ldots, n). \tag{33}$$

This result indicates that the energy norms are locally monotonically decreasing functions of the trial frequency.

However, on the curve of d_i there may be some singular points at which the value of d_i is not continuous. These singular points are associated with the trial values $\bar{\lambda}_{\mathrm{s}}$ where $D_{i-1}(\bar{\lambda}_{\mathrm{s}}) = 0$ but $D_i(\bar{\lambda}_{\mathrm{s}}) \neq 0$. As the first step of the proof, a function $J^{(m)}(\bar{\lambda})$, called the partial sign count associated with $\mathbf{K}_{\mathrm{D}}^{(m)}(\bar{\lambda})$, the mth principal sub-matrix of $\mathbf{K}_{\mathrm{D}}(\bar{\lambda})$, is defined by

$$J^{(m)}(\bar{\lambda}) = \sum_{k=1}^{n_{\mathrm{s}}} J_k(\bar{\lambda}) + s\{\mathbf{K}_{\mathrm{D}}^{(m)}(\bar{\lambda})\} \quad (m = 1, \ldots, n). \tag{34}$$

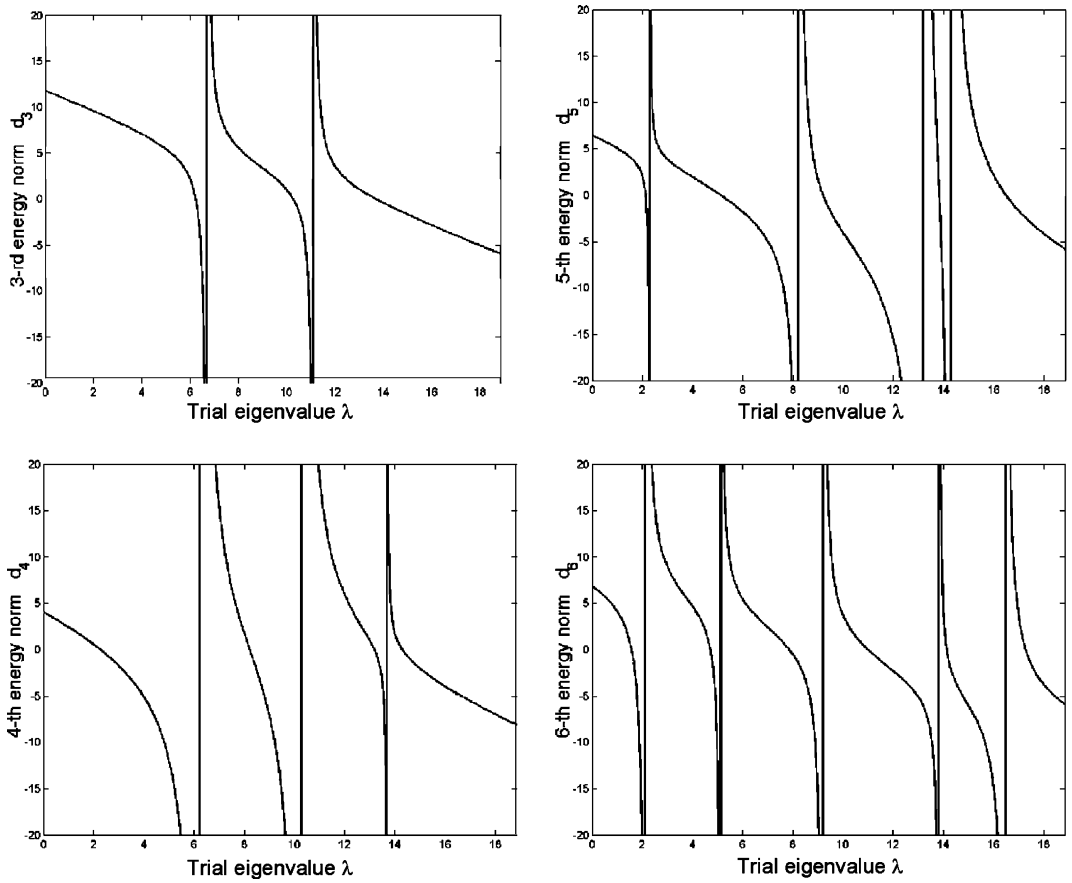

Fig. 3. Curves of energy norms.

When the trial value $\bar{\lambda}$ is changed from $\bar{\lambda}_s - \varepsilon$ to $\bar{\lambda}_s + \varepsilon$, where ε is a small enough positive real number, the value of $J^{(i-1)}(\bar{\lambda})$ increases by one. Meanwhile, $D_i(\bar{\lambda}_s) \neq 0$ requires that the value of $J^{(i)}(\bar{\lambda})$ is invariant. Therefore, from Eq. (24) it is seen that d_i must jump from $-\infty$ to $+\infty$ at $\bar{\lambda}_s$. Consequently, each curve of the energy norms is generally separated into a number of branches by some singular points, and in each of these branches the energy norm is a monotonically decreasing function of the trial frequency, as illustrated by Fig. 3.

5. Criterion for eigenfrequencies in terms of energy norms

From a physical point of view, the sequence of displacement vectors that can reduce the dynamic stiffness matrix into diagonal form is one that gradually approaches the modes of vibration, in the sense that fewer and fewer non-zero forces are needed to support the corresponding displacements. For the last mode of displacements $\mathbf{p}_n$, at most only one non-zero force, equal to the last energy norm d_n, is necessary. It is obvious that if a value of λ is found such that

$$d_n(\lambda) = 0, \tag{35}$$

then λ is definitely an eigenvalue and $\mathbf{p}_n$ is the corresponding mode of vibration.

In practice, most of the eigenvalues satisfy Eq. (35). However, it is theoretically possible that some eigenvalues would be omitted if Eq. (35) were taken as the criterion. An obvious such case occurs when the dynamic stiffness matrix is diagonal and each of its elements is a linear function of λ. In this case, λ is an eigenvalue if any of the $d_i(\lambda)$ $(i = 1, \ldots, n)$ is equal to zero. In general, Eq. (35) will not be satisfied for any modes of vibration for which the last degree of freedom is nodal, i.e. its displacement is zero.

However such exceptional cases are not an insurmountable obstacle, since any degree of freedom can be re-numbered to be the last one. The question of how to choose the appropriate degree of freedom to be the last one is now addressed.

When the ith eigenvalue λ_i is being located, the bounds on λ_i can be categorized into local and proper ones, as illustrated in Fig. 4. A pair of bounds $\bar{\lambda}_i^{\mathrm{l}}$ and $\bar{\lambda}_i^{\mathrm{u}}$ are called local if λ_i is the unique eigenvalue that lies within the interval $(\bar{\lambda}_i^{\mathrm{l}}, \bar{\lambda}_i^{\mathrm{u}})$. If also there are no poles in this interval, so that $\bar{\lambda}_i^{\mathrm{l}}$ and $\bar{\lambda}_i^{\mathrm{u}}$ lie on the same branch of the curve of the last energy norm, then they are called proper bounds. Local and proper bounds can be recognized by evaluating the corresponding sign counts. Hence if $\bar{\lambda}_i^{\mathrm{l}}$ and $\bar{\lambda}_i^{\mathrm{u}}$ are found to be such that

$$J(\bar{\lambda}_i^{\mathrm{l}}) = i - 1 \qquad \text{and} \qquad J(\bar{\lambda}_i^{\mathrm{u}}) = i, \tag{36}$$

they are definitely at least local bounds on λ_i. Then, if $d_n(\bar{\lambda}_i^{\mathrm{l}})$ is positive and $d_n(\bar{\lambda}_i^{\mathrm{u}})$ is negative, $\bar{\lambda}_i^{\mathrm{l}}$ and $\bar{\lambda}_i^{\mathrm{u}}$ are proper bounds and the zero point on the curve of $d_n(\bar{\lambda})$ in the interval $(\bar{\lambda}_i^{\mathrm{l}}, \bar{\lambda}_i^{\mathrm{u}})$ must be the eigenvalue λ_i. Unlike the local bounds, the proper bounds depend on the last energy norm. When $\bar{\lambda}_i^{\mathrm{l}}$ and $\bar{\lambda}_i^{\mathrm{u}}$ are a pair of local bounds but are not proper bounds for $d_n(\bar{\lambda})$, they will be a pair of proper bounds for the last energy norm $d_n^{(j)}(\bar{\lambda})$, which denotes that the jth degree of freedom has been re-numbered as the last one, as long as they satisfy

$$d_n^{(j)}(\bar{\lambda}_i^{\mathrm{l}}) > 0 \qquad \text{and} \qquad d_n^{(j)}(\bar{\lambda}_i^{\mathrm{u}}) < 0. \tag{37}$$

Consequently, if a pair of local bounds has been found, the last degree of freedom can be chosen from those that satisfy Eq. (37), as illustrated in Fig. 5.

In the process of choosing the last degree of freedom, the values of $d_n^{(j)}(\bar{\lambda})$ $(j = 1, \ldots, n)$ are needed. These can be obtained by a minor additional calculation after computation of the energy norms $d_i(\bar{\lambda})$ $(i = 1, \ldots, n)$ corresponding to the original order of the freedoms, due to their simple relationship to the inverse of the dynamic stiffness matrix, i.e.

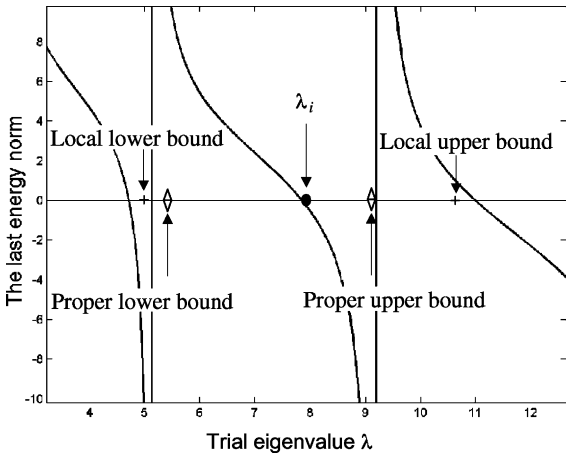

Fig. 4. Definition of local and proper bounds.

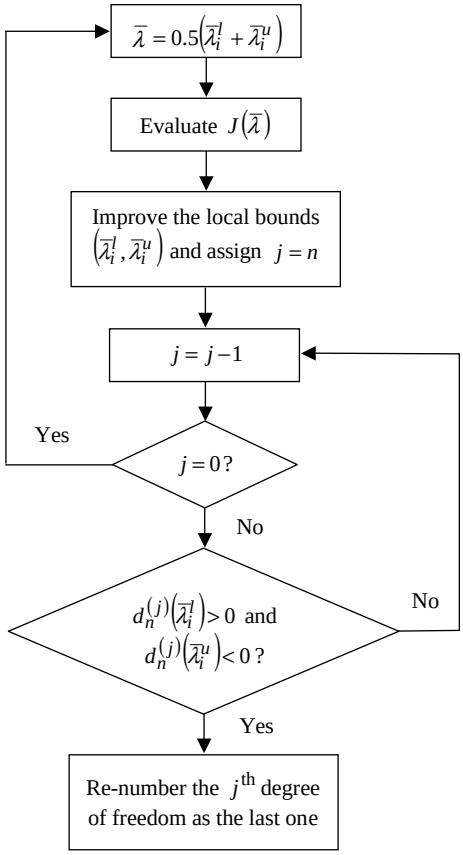

Fig. 5. Procedure for choosing the last degree of freedom.

$$d_n^{(j)} = \frac{1}{(K_{\mathrm{D}}^{-1})_{jj}} \tag{38}$$

where $(K_{\mathrm{D}}^{-1})_{jj}$ is the jth diagonal element of the inverse of the matrix $\mathbf{K}_{\mathrm{D}}(\bar{\lambda})$. This relationship results from the physical interpretation of the jth column of the inverse of the dynamic stiffness matrix as being the mode of displacements generated by applying a unit force at the jth degree of freedom only, so that $(K_{\mathrm{D}}^{-1})_{jj}$ is the corresponding displacement at the jth degree of freedom. From this interpretation, it is easily seen that the force needed to produce unit displacement at the jth degree of freedom is simply the right-hand side of Eq. (38).

From Eq. (17), the inverse of the dynamic stiffness matrix can be expressed in terms of the modes of displacements as

$$\mathbf{K}_{\mathrm{D}}^{-1}(\bar{\lambda}) = \mathbf{P}\mathbf{D}^{-1}\mathbf{P}^{\mathrm{T}}. \tag{39}$$

Then combining Eqs. (38) and (39) and remembering that $\mathbf{P}$ is upper triangular gives

$$d_n^{(j)} = \frac{1}{\sum_{k=j}^{n} \dfrac{p_{kj}^2}{d_k}}, \tag{40}$$

where the jth component of the ith mode of displacement is denoted as p_{ij}. When the jth degree of freedom is re-numbered as the last one, the corresponding last mode of displacements changes from $\mathbf{p}_n$ to $\mathbf{p}_n^{(j)}$, which can also be expressed in terms of the original modes of displacements as

$$p_{ni}^{(j)} = \frac{\sum_{k=\max(i,j)}^{n} \frac{p_{ki}p_{kj}}{d_k}}{\sum_{k=j}^{n} \frac{p_{kj}^2}{d_k}} \quad (i = 1, \ldots, n), \tag{41}$$

where $p_{ni}^{(j)}$ is the ith component of $\mathbf{p}_n^{(j)}$. This result comes from the normalization of the jth column of the inverse of $\mathbf{K}_{\mathrm{D}}(\bar{\lambda})$ by making its jth component unity while keeping the ratios of it to the rest of the components invariant.

It is seen from the expressions in Eqs. (40) and (41) that the computations to obtain $d_n^{(j)}(\bar{\lambda})$ and $\mathbf{p}_n^{(j)}$ are quite simple and can be completed efficiently. Therefore, when combined with the appropriate choice of the last energy norm, Eq. (35) is a suitable criterion for obtaining eigenfrequencies. Moreover, it is also a synthetic criterion in the sense that both the eigenfrequencies and modes of vibration can be obtained highly accurately if Eq. (35) is approximated to high accuracy.

Another question about the criterion, for which theory is necessary even though it rarely occurs in practice, is what to do when a mode of vibration of interest happens to be such that the values of all of the visible variables are zero, i.e. $\mathbf{x}_c = \mathbf{0}$. (This possibility was recognized and dealt with in the first (Williams and Wittrick, 1970) paper on the Wittrick–Williams algorithm.) It seems impossible to employ Eq. (35) as the criterion because obviously it cannot be satisfied no matter which degree of freedom is re-numbered as the last one. However, the difficulty can obviously be easily overcome by changing one or more of the invisible variables $\mathbf{x}_i$ into visible ones.

Without loss of generality, suppose that just one invisible variable u becomes a visible one. As the invisible variable is always associated with an inner node of a sub-structure, the corresponding degree of freedom is ordered prior to any of the visible variables. Therefore, the energy norm d_0 and the mode of displacement $\mathbf{p}_0$ corresponding to the selected invisible variable u can be obtained through the dynamic stiffness matrix for the sub-structure. For example, if $\overline{\mathbf{K}}_{\mathrm{D}}(\bar{\lambda}, \rho)$ is the known dynamic stiffness matrix for a beam of length ρ (including axial behaviour) and is expressed as

$$\overline{\mathbf{K}}_{\mathrm{D}}(\bar{\lambda}, \rho) = \begin{bmatrix} \overline{\mathbf{K}}_{\mathrm{D}}^{11}(\bar{\lambda}, \rho) & \overline{\mathbf{K}}_{\mathrm{D}}^{12}(\bar{\lambda}, \rho) \\ \overline{\mathbf{K}}_{\mathrm{D}}^{21}(\bar{\lambda}, \rho) & \overline{\mathbf{K}}_{\mathrm{D}}^{22}(\bar{\lambda}, \rho) \end{bmatrix}, \tag{42}$$

and if u is chosen as the kth displacement ($k = 1, 2$ or 3) at the point with co-ordinate ξ, see Fig. 6, then, based on its physical interpretation, the corresponding energy norm d_0 can be written as

$$d_0 = \mathbf{e}_k^{\mathrm{T}} \overline{\mathbf{K}}_{\mathrm{D}}^{22}(\bar{\lambda}, \xi)\mathbf{e}_k + \mathbf{e}_k^{\mathrm{T}} \overline{\mathbf{K}}_{\mathrm{D}}^{11}(\bar{\lambda}, \rho - \xi)\mathbf{e}_k, \tag{43}$$

where $\mathbf{e}_k$ is the kth column of the unit matrix of order 3. The mode of displacement $\mathbf{p}_0$ is the first column of the unit matrix of order $n + 1$.

When u is re-numbered as the last degree of freedom, it is seen from Eqs. (40) and (41) that the corresponding energy norm $d_n^{(0)}$ and the components of the mode of displacements can be written as

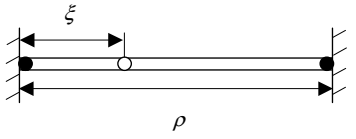

Fig. 6. Beam of length ρ, showing (with a hollow circle) an inner node associated with an invisible variable.

$$d_n^{(0)} = \frac{d_0}{1 + \sum_{k=1}^{n} \frac{d_0}{d_k} p_{k0}^2} \tag{44}$$

$$p_{ni}^{(0)} = \frac{\sum_{k=i}^{n} \frac{d_0}{d_k} p_{ki} p_{k0}}{1 + \sum_{k=1}^{n} \frac{d_0}{d_k} p_{k0}^2} \quad (i = 1, \ldots, n), \qquad p_{n0}^{(0)} = 1, \tag{45}$$

where p_{k0} is defined as the value of u when the original visible variables are equal to the components of $\mathbf{p}_k$ respectively under the action of forces $\mathbf{f}_i$ defined by Eq. (22). The value of u can be assigned by the exact expressions of displacements in a sub-structure in terms of the displacements at its boundary points. For example, the equations

$$u(x) = \left(\cos\frac{vx}{l} - \cot v \sin\frac{vx}{l} \right) u(0) + \operatorname{cosec} v \sin\frac{vx}{l} u(l), \qquad v = l\sqrt{\frac{\mu\omega^2}{\mathrm{EA}}} \tag{46}$$

express the axial displacement of a bar in terms of its boundary displacements $u(0)$ and $u(l)$, which are always visible variables and so are among the components of the mode of displacements of the structure. When the axial displacement at $x = \xi$ is chosen as a new visible variable, the value of p_{k0} is obtained by calculating $u(\xi)$ with the boundary displacements extracted from $\mathbf{p}_k$. If the analytical expression is inconvenient, the value of p_{k0} can be obtained by applying Eq. (7) where $\mathbf{x}_c$ is the vector of the boundary displacements and the single element of $\mathbf{x}_i$ is the value of the new visible variable. The result may be analytical, if the analytical dynamic stiffness matrix is employed, or approximate if approximate matrices are used.

It is important to note that proper bounds on a specific eigenvalue are always found during the process of recognizing the suitable last degree of freedom.

6. Computational method for the eigensolution

Since d_n is a monotonically decreasing function of $\bar{\lambda}$ in the interval $(\bar{\lambda}_i^{\mathrm{l}}, \bar{\lambda}_i^{\mathrm{u}})$ bounded by the proper bounds on an eigenvalue λ_i, the map from $\bar{\lambda}$ to d_n is one to one. Hence $\bar{\lambda}$ is a well-defined function of d_n. By inverting the right-hand side of Eq. (31), the derivative of this function with respect to d_n is obtained as

$$\frac{\mathrm{d}\bar{\lambda}}{\mathrm{d}d_n} = \frac{1}{\mathbf{p}_n^{\mathrm{T}} \frac{\mathrm{d}\mathbf{K}_{\mathrm{D}}(\bar{\lambda})}{\mathrm{d}\bar{\lambda}} \mathbf{p}_n}. \tag{47}$$

The derivative defined by Eq. (47) is denoted as $f(\bar{\lambda})$ in order to simplify the notation. When a trial value $\bar{\lambda}_0$ is chosen within the interval $(\bar{\lambda}_i^{\mathrm{l}}, \bar{\lambda}_i^{\mathrm{u}})$, the point with co-ordinates $(\bar{\lambda}_0, d_n(\bar{\lambda}_0))$ must lie on the branch of the curve that has its unique zero point located where $\bar{\lambda} = \lambda_i$ and $d_n(\bar{\lambda}) = 0$. Therefore, the eigenproblem can be transformed to the initial value problem constituted by the ordinary differential equations (47) and the initial point $(\bar{\lambda}_0, d_n(\bar{\lambda}_0))$, in which the eigenvalue λ_i can be found by locating the point $(\lambda_i, 0)$ on the curve. This transformation is very important, because it enables a variety of highly developed methods of numerical analysis for ordinary differential equations to be applied to obtain the eigensolutions. For example, if the one-step Euler method is employed, the approximated eigenvalue is given by

$$\lambda = \bar{\lambda}_0 + hf(\bar{\lambda}_0) = \bar{\lambda}_0 - \frac{\mathbf{p}_n^{\mathrm{T}} \mathbf{K}_{\mathrm{D}}(\bar{\lambda}_0) \mathbf{p}_n}{\mathbf{p}_n^{\mathrm{T}} \frac{\mathrm{d}\mathbf{K}_{\mathrm{D}}(\bar{\lambda}_0)}{\mathrm{d}\bar{\lambda}} \mathbf{p}_n}, \tag{48}$$

where

$$\frac{\mathrm{d}\mathbf{K}_{\mathrm{D}}(\bar{\lambda}_0)}{\mathrm{d}\bar{\lambda}} = \left.\frac{\mathrm{d}\mathbf{K}_{\mathrm{D}}(\bar{\lambda})}{\mathrm{d}\bar{\lambda}}\right|_{\bar{\lambda}=\bar{\lambda}_0} \tag{49}$$

and

$$h = -d_n(\bar{\lambda}_0) \tag{50}$$

is the time step for integration. When a linear dynamic stiffness matrix, e.g. $\mathbf{K}_{\mathrm{D}}(\bar{\lambda}) = \mathbf{K} - \bar{\lambda}\mathbf{M}$, is used, the differential of $\mathbf{K}_{\mathrm{D}}(\bar{\lambda})$ is reduced to

$$\frac{\mathrm{d}\mathbf{K}_{\mathrm{D}}(\bar{\lambda}_0)}{\mathrm{d}\bar{\lambda}} = -\mathbf{M} \tag{51}$$

and Eq. (48) gives

$$\lambda = \frac{\mathbf{p}_n^{\mathrm{T}}\mathbf{K}\mathbf{p}_n}{\mathbf{p}_n^{\mathrm{T}}\mathbf{M}\mathbf{p}_n}, \tag{52}$$

which, interestingly, is simply Rayleigh's quotient. Since Rayleigh's quotients, which remain a powerful tool when obtaining eigenvalues in structural dynamics, can be regarded as the simplest method of integration for solving initial value problems, it is reasonable to expect that better results can be obtained by using more accurate methods of integration, as follows.

The one-step fourth order Runge–Kutta method is preferred in this paper, because of its high accuracy and simplicity. It estimates the eigenvalue in the following way:

$$\lambda = \bar{\lambda}_0 + [\tfrac{1}{6} f(\bar{\lambda}_0) + \tfrac{1}{3} f(\bar{\lambda}_1) + \tfrac{1}{3} f(\bar{\lambda}_2) + \tfrac{1}{6} f(\bar{\lambda}_3)]h, \tag{53}$$

where

$$\bar{\lambda}_1 = \bar{\lambda}_0 + \tfrac{1}{2} f(\bar{\lambda}_0)h, \tag{54}$$

$$\bar{\lambda}_2 = \bar{\lambda}_0 + \tfrac{1}{2} f(\bar{\lambda}_1)h, \tag{55}$$

$$\bar{\lambda}_3 = \bar{\lambda}_0 + f(\bar{\lambda}_2)h. \tag{56}$$

Once an approximated eigenvalue λ is known, the absolute value of $d_n(\lambda)$ is checked to see whether or not it is larger than a given tolerance. When it is not, Eq. (53) is repeated with $\bar{\lambda}_0 = \lambda$ and $h = -d_n(\lambda)$. Moreover, the bounds on the required eigenvalues λ_i are continuously improved during the evaluation of the derivatives $f(\bar{\lambda}_k)$ $(k = 0, 1, 2, 3)$ in Eq. (53) according to the following principle: when $s\{\mathbf{K}_{\mathrm{D}}(\bar{\lambda})\} = i$, $\bar{\lambda}$ is a better proper lower bound on λ_{i+1} if $\bar{\lambda} > \bar{\lambda}_{i+1}^{\mathrm{l}}$, or a better proper upper bound on λ_i if $\bar{\lambda} < \bar{\lambda}_i^{\mathrm{u}}$.

If the derivatives in Eq. (53) are not from the same branch, the estimated eigenvalue may be worse than the initial value $\bar{\lambda}_0$. However, it is observed in practice that such an unfavourable situation rarely occurs when the initial value $\bar{\lambda}_0$ lies between the proper bounds. Nevertheless, there is no strict proof to show that such a situation can never occur and so the following stabilization control is highly recommended. When the ith eigenfrequency is being sought, all of the $f(\bar{\lambda}_k)$ $(k = 0, 1, 2, 3)$ are obviously from the same curve if all the trial values $\bar{\lambda}_0$, $\bar{\lambda}_1$, $\bar{\lambda}_2$ and $\bar{\lambda}_3$ lie between the proper bounds $\bar{\lambda}_i^{\mathrm{l}}$ and $\bar{\lambda}_i^{\mathrm{u}}$. This condition amounts to the following requirements:

$$\left.\begin{array}{ll} \text{Either} & [s\{\mathbf{K}_{\mathrm{D}}(\bar{\lambda}_k)\} = i-1 \text{ and } d_n(\bar{\lambda}_k) > 0] \\ \text{or} & [s\{\mathbf{K}_{\mathrm{D}}(\bar{\lambda}_k)\} = i \text{ and } d_n(\bar{\lambda}_k) < 0] \end{array}\right\} \quad (k = 0, 1, 2, 3). \tag{57}$$

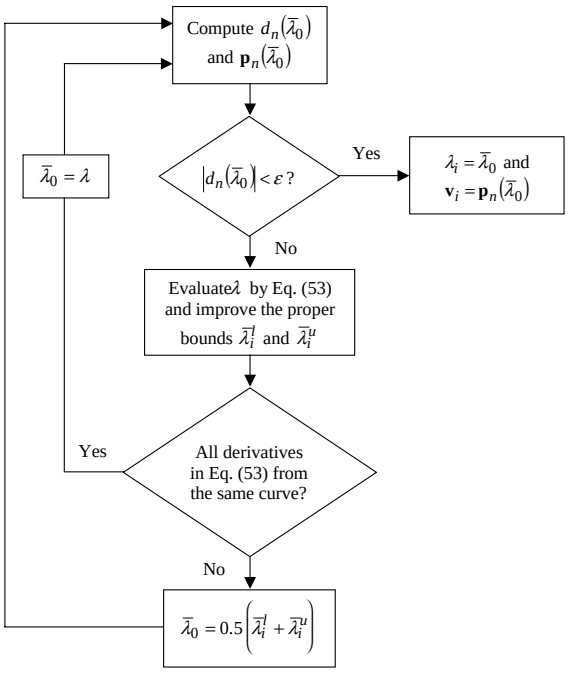

Fig. 7. Procedure for finding eigenvalue λ_i and its mode of vibration $\mathbf{v}_i$.

If these requirements are not met, the long established bisection method is used to obtain a new trial value

$$\bar{\lambda} = \frac{\bar{\lambda}_i^{\mathrm{l}} + \bar{\lambda}_i^{\mathrm{u}}}{2} \tag{58}$$

which becomes the initial value for the next iteration. Despite its low accuracy, Eq. (58) gives a better estimate of the eigenvalue than does $\bar{\lambda}_0$. The process of finding a specific eigenvalue λ_i and its associated mode of vibration $\mathbf{v}_i$ is illustrated in Fig. 7.

It seems that the differential of the dynamic stiffness matrix should be assembled in order to calculate the derivatives in Eq. (53). However, only the differential of $\mathbf{K}_{\mathrm{D}}(\bar{\lambda})$ for each member of the structure is required because the denominator on the right-hand side of Eq. (47) can be obtained by adding its components from each member of the structure, i.e.

$$\mathbf{p}_n^{\mathrm{T}} \frac{\mathrm{d}\mathbf{K}_{\mathrm{D}}}{\mathrm{d}\bar{\lambda}} \mathbf{p}_n = \sum_{i=1}^{n_{\mathrm{c}}} \bar{\mathbf{p}}_i^{\mathrm{T}} \frac{\mathrm{d}\overline{\mathbf{K}}_{\mathrm{D}}^i}{\mathrm{d}\bar{\lambda}} \bar{\mathbf{p}}_i, \tag{59}$$

where $\bar{\mathbf{p}}_i$ is the vector of displacements of the ith member resulting from the mode of displacements $\mathbf{p}_n$, $\overline{\mathbf{K}}_{\mathrm{D}}^i$ is the dynamic stiffness matrix of the ith member and n_{c} is the number of members.

The differential of the dynamic stiffness matrix for a Bernoulli–Euler beam with axial displacements can be obtained from the equations

$$\frac{\mathrm{d}a}{\mathrm{d}\lambda} = \frac{1}{2\lambda}\left(a - \frac{l}{\mathrm{EA}} e^2\right), \qquad \frac{\mathrm{d}e}{\mathrm{d}\lambda} = \frac{1}{2\lambda}\left(e - \frac{l}{\mathrm{EA}} ae\right), \tag{60}$$

$$\frac{\mathrm{d}b}{\mathrm{d}\lambda} = \frac{1}{4\lambda}\left(3b - \frac{l}{\mathrm{EI}} h^2\right), \qquad \frac{\mathrm{d}f}{\mathrm{d}\lambda} = \frac{1}{4\lambda}\left(3f + \frac{l}{\mathrm{EI}}\,\mathrm{d}h\right), \tag{61}$$

$$\frac{\mathrm{d}c}{\mathrm{d}\lambda}=\frac{1}{4\lambda}\left(c+2dl-\frac{l}{\mathrm{EI}}c^2\right),\qquad \frac{\mathrm{d}g}{\mathrm{d}\lambda}=\frac{1}{4\lambda}\left(g+hl-\frac{l}{\mathrm{EI}}cg\right), \tag{62}$$

$$\frac{\mathrm{d}d}{\mathrm{d}\lambda}=\frac{1}{4\lambda}\left(2d-\frac{l}{\mathrm{EI}}gh\right),\qquad \frac{\mathrm{d}h}{\mathrm{d}\lambda}=\frac{1}{4\lambda}\left(2h-\frac{l}{\mathrm{EI}}gd\right), \tag{63}$$

where $\lambda=\omega^2$ and a, b, c, d, e, f, g and h are the independent values in the dynamic stiffness matrix for a Bernoulli–Euler member with uncoupled axial behaviour, i.e.

$$\mathbf{K}_{\mathrm{D}}(\lambda)=\begin{bmatrix} a & 0 & 0 & e & 0 & 0\\ 0 & b & d & 0 & f & h\\ 0 & d & c & 0 & -h & g\\ e & 0 & 0 & a & 0 & 0\\ 0 & f & -h & 0 & b & -d\\ 0 & h & g & 0 & -d & c \end{bmatrix}, \tag{64}$$

where (Kolousek, 1973; Howson, 1979):

$$a=\frac{\mathrm{EA}\,v}{l}\cot v,\qquad e=-\frac{\mathrm{EA}\,v}{l}\operatorname{cosec} v, \tag{65}$$

$$b=\frac{\mathrm{EI}\,\eta^3}{Zl^3}(\sin\eta\cosh\eta+\cos\eta\sinh\eta),\qquad f=-\frac{\mathrm{EI}\,\eta^3}{Zl^3}(\sin\eta+\sinh\eta), \tag{66}$$

$$c=\frac{\mathrm{EI}\,\eta}{Zl}(\sin\eta\cosh\eta-\cos\eta\sinh\eta),\qquad g=\frac{\mathrm{EI}\,\eta}{Zl}(\sinh\eta-\sin\eta), \tag{67}$$

$$d=\frac{\mathrm{EI}\,\eta^2}{Zl^2}(\sin\eta\sinh\eta),\qquad h=\frac{\mathrm{EI}\,\eta^2}{Zl^2}(\cosh\eta-\cos\eta), \tag{68}$$

$$v=l\sqrt{\frac{\mu\omega^2}{\mathrm{EA}}},\qquad \eta=l\sqrt[4]{\frac{\mu\omega^2}{\mathrm{EI}}},\qquad Z=1-\cos\eta\cosh\eta \tag{69}$$

and l, μ, EA and EI are, respectively, the length, mass per unit length, extensional rigidity and flexural rigidity of the member.

7. Numerical example

In order to make a comparison between the bisection method and the method proposed in this paper, a problem used by Howson (1979) was solved. This problem was to find those eigenfrequencies $\sqrt{\lambda_i}$ and modes of vibration of the frame shown in Fig. 8 that are listed in Tables 1–3. All the beams are made of the same material and have identical cross-sections, with flexural rigidity $\mathrm{EI}=5\ \mathrm{MNm}^2$, extensional rigidity $\mathrm{EA}=900\ \mathrm{MN}$ and mass per unit length $\mu=35\ \mathrm{kgm}^{-1}$. The frame is fully built-in at node 1 and horizontal movement is prevented at node 2.

In Howson (1979), the tolerance for the relative difference between the upper and lower bounds on the eigenfrequencies was 10^{-6}. In the present paper, this value of tolerance is instead used as an absolute tolerance on the last energy norm d_n. The results are shown in Table 1, where ε_1 is the largest absolute value of any of the forces associated with the corresponding mode of vibration. The smaller ε_1 is, the more accurate the corresponding mode of vibration should be.

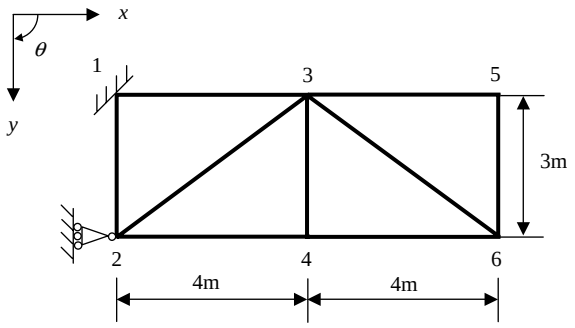

Fig. 8. A frame composed of beams.

Table 1
Comparison between the proposed method and the bisection method, showing eigenfrequencies $\sqrt{\lambda_i}$ (in Hz), the last energy norm d_n and measure of mode accuracy ε_1

i	Proposed method			Bisection method	
	$\sqrt{\lambda_i}$	d_n	ε_1	$\sqrt{\lambda_i}$	ε_1
1	25.4676198	4.67922169×10^{-8}	8.96980055×10^{-8}	25.4676200	0.05709290
2	39.4660333	4.20521418×10^{-8}	4.20259312×10^{-8}	39.4660334	0.19144716
3	49.1171943	5.67928732×10^{-7}	2.68220901×10^{-7}	49.1171943	0.02453265
20	256.203861	-8.98726285×10^{-7}	8.94069672×10^{-7}	256.203861	3.28365721

Table 2
Comparison of speed of convergence

Eigenfrequency number	Number of iterations by proposed method	Number of iterations by bisection method
1	8	25
2	12	25
3	8	24
20	4	26

Table 3
History of the last energy norm d_n and measure of mode accuracy ε_1 versus iteration number of the integration

Eigenfrequency number	Iteration number	d_n	ε_1
1	1	1.10085758×10^{-1}	1.10085785×10^{-1}
	2	4.67922169×10^{-8}	8.96980055×10^{-8}
2	1	-5.25669286×10^{5}	5.25669286×10^{-2}
	2	-6.31791522×10^{-2}	6.31791531×10^{-2}
	3	4.20521418×10^{-8}	4.20259312×10^{-8}
3	1	1.74112694×10^{2}	7.78778037×10^{1}
	2	5.67928732×10^{-7}	2.68220901×10^{-7}
20	1	-8.98726285×10^{-7}	8.94069672×10^{-7}

It is seen from the results that the slight differences between the eigenfrequencies obtained are within the specified absolute tolerance of 10^{-6}, but that the accuracy of the modes of vibration is improved greatly by the method of the present paper. Moreover, the eigenfrequencies obtained by the present method are more

accurate than those obtained by the bisection method. This conclusion was reached by repeating the bisection results with a smaller tolerance.

In order to compare the speed of convergence, Table 2 shows the number of iterations needed to make the last energy norm smaller than the given tolerance for the present method, and to make the separation of the bounds on the eigenfrequency smaller than the same tolerance for the bisection method. This number shows how many times it is necessary to reduce the dynamic stiffness matrix to diagonal form before the given accuracy is obtained, noting that to make the comparison meaningful the iteration counts began only after the same proper bounds had been found for each method. The results shown in Table 2 indicate that the proposed method is much more efficient than bisection. Because each iteration of Runge–Kutta integration requires four such reductions of the dynamic stiffness matrix, Table 2 indicates that most eigenfrequencies and modes of vibration of structures are likely to be found within three iterations of Runge–Kutta integration. Finally, the values of d_n and ε_1 for each iteration of the integration are shown in Table 3, illustrating rapid convergence of both the eigenfrequencies and the modes of vibration. Therefore, the method proposed in this paper is very efficient.

8. Conclusions

The method presented in this paper is an efficient and highly accurate method for computing the eigensolutions of transcendental eigenproblems arising in structural dynamics. It is based on a new criterion for eigenfrequency. Instead of trying to find the roots of the determinant of the dynamic stiffness matrix, the method seeks the frequency that makes the last energy norm vanish, which is usually a physical definition of the eigenfrequency. Rules have been presented for interchanging degrees of freedom to obtain an alternative last energy norm for the exceptional problems for which the original last energy norm corresponds to a displacement which is nodal in the mode. By the property of energy norms, the eigenproblem is transformed into a specific initial value problem of an ordinary differential equation. Among many available methods to solve the resulting ordinary differential equation, the one-step Runge–Kutta method is proved to be a simple and efficient method to obtain eigensolutions, as confirmed by the numerical example.

Acknowledgements

This work was supported by the China Association for International Exchange of Personnel. The first author is especially grateful for the financial support of Cardiff University and the National Science Foundation of China. The third author holds a chair at Cardiff University to which he will return upon completion of his appointment at City University of Hong Kong.

References

Bathe, K.J., 1996. Finite Element Procedures. Prentice-Hall, Englewood Cliffs, NJ.

Blaszkowiak, S., Kaczkowski, Z., 1966. Iterative Methods in Structural Analysis. Pergamon, Oxford.

Collar, A.R., Simpson, A., 1987. Matrices and Engineering Dynamics. Ellis Horwood, Chichester.

Hager, P., Wiberg, N.E., 1999. Adaptive eigenfrequency analysis by superconvergent patch recovery. Computer Methods in Applied Mechanics and Engineering 176 (1–4), 441–462.

Hopper, C.T., Williams, F.W., 1977. Mode finding in nonlinear structural eigenvalue calculations. Journal of Structural Mechanics 5 (3), 255–278.

Howson, W.P., 1979. A compact method for computing the eigenvalues and eigenvectors of plane frames. Advances in Engineering Software 1 (4), 181–190.

Howson, W.P., Banerjee, J.R., Williams, F.W., 1983. Concise equations and program for exact eigensolutions of plane frames including member shear. Advances in Engineering Software 5 (3), 137–141.

Kennedy, D., Williams, F.W., 1991. More efficient use of determinants to solve transcendental structural eigenvalue problems reliably. Computers and Structures 41 (5), 973–979.

Kolousek, V., 1973. Dynamics in Engineering Structures. Butterworths, London.

Meirovitch, L., 1980. Computational Methods in Structural Dynamics. Sijthoff and Noordhoff, Netherlands.

Mirsky, L., 1982. An Introduction to Linear Algebra. Constable, London.

Ronagh, H.R., Lawther, R., Williams, F.W., 1995. Calculation of eigenvectors with uniform accuracy. ASCE Journal of Engineering Mechanics 121 (9), 948–955.

Simpson, A., 1984. On the solution of $\mathbf{S}(\omega)\mathbf{x} = \mathbf{0}$ by a Newtonian procedure. Journal of Sound and Vibration 97 (1), 153–164.

Wiberg, N.E., Bausys, R., Hager, P., 1999. Adaptive h-version eigenfrequency analysis. Computers and Structures 71 (5), 565–584.

Williams, F.W., Anderson, M.S., 1986. Inclusion of elastically connected members in exact buckling and frequency calculations. Computers and Structures 22 (3), 395–397.

Williams, F.W., Kennedy, D., 1988. Reliable use of determinants to solve non-linear structural eigenvalue problems efficiently. International Journal for Numerical Methods in Engineering 26 (8), 1825–1841.

Williams, F.W., Wittrick, W.H., 1970. An automatic computational procedure for calculating natural frequencies of skeletal structures. International Journal of Mechanical Sciences 12 (9), 781–791.

Williams, F.W., Wittrick, W.H., 1983. Exact buckling and frequency calculations surveyed. ASCE Journal of Structural Engineering 109 (1), 169–187.

Williams, F.W., Yuan, S., 1999. Calculating the modes from dynamic stiffness matrix analysis of piecewise continuous structures. In: Proceedings of EPMESC VII: International Conference on Enhancement and Promotion of Computational Methods in Engineering and Science, Macao, 1. Elsevier, Oxford, pp. 89–99.

Wittrick, W.H., Williams, F.W., 1971. A general algorithm for computing natural frequencies of elastic structures. Quarterly Journal of Mechanics and Applied Mathematics 24 (3), 263–284.

Wittrick, W.H., Williams, F.W., 1973. An algorithm for computing critical buckling loads of elastic structures. Journal of Structural Mechanics 1 (4), 497–518.

Ye, J., Williams, F.W., 1995. A successive bounding method to find the exact eigenvalues of transcendental stiffness matrix formulations. International Journal for Numerical Methods in Engineering 38 (6), 1057–1067.